Local and global existence theorems for the Einstein equations

Alan D. Rendall, Max-Planck-Institut für Gravitationsphysik,
Am Mühlenberg 1, 14476 Golm, Germany,
rendall@aei-potsdam.mpg.de

Abstract

This article is a guide to the literature on existence theorems for the Einstein equations which also draws attention to open problems in the field. The local in time Cauchy problem, which is relatively well understood, is treated first. Next global results for solutions with symmetry are discussed. A selection of results from Newtonian theory and special relativity which offer useful comparisons is presented. This is followed by a survey of global results in the case of small data and results on constructing spacetimes with given singularity structure. The article ends with some miscellaneous topics connected with the main theme.

1 Introduction

Many of the mathematical models occurring in physics involve systems of partial differential equations. Only rarely can these equations be solved by explicit formulae. When they cannot, physicists frequently resort to approximations. There is, however, another approach which is complementary. This consists in determining the qualitative behaviour of solutions, without knowing them explicitly. The first and most fundamental step in doing this is to establish the existence of solutions under appropriate circumstances. Unfortunately, this is often hard, and obstructs the way to obtaining more interesting information. It may appear to the outside observer that existence theorems become a goal in themselves to some researchers. It is important to remember that, from a more general point of view, they are only a first step.

The basic partial differential equations of general relativity are Einstein's equations. In general they are coupled to other partial differential equations describing the matter content of spacetime. The Einstein equations are essentially hyperbolic in nature. In other words, the general properties of solutions are similar to those found for the wave equation. It follows that it is reasonable to try to determine a solution by initial data on a spacelike hypersurface. Thus the

Cauchy problem is the natural context for existence theorems for the Einstein equations. The Einstein equations are also nonlinear. This means that there is a big difference between the local and global Cauchy problems. A solution evolving from regular data may develop singularities.

A special feature of the Einstein equations is that they are diffeomorphism invariant. If the equations are written down in an arbitrary coordinate system then the solutions of these coordinate equations are not uniquely determined by initial data. Applying a diffeomorphism to one solution gives another solution. If this diffeomorphism is the identity on the chosen Cauchy surface up to first order then the data are left unchanged by this transformation. In order to obtain a system for which uniqueness in the Cauchy problem holds in the straightforward sense it does for the wave equation, some coordinate or gauge fixing must be carried out.

Another special feature of the Einstein equations is that initial data cannot be given freely. They must satisfy constraint equations. To prove the existence of a solution of the Einstein equations, it is first necessary to prove the existence of a solution of the constraints. The usual method of solving the constraints relies on the theory of elliptic equations.

The local existence theory of solutions of the Einstein equations is rather well understood. Section 2 points out some of the things which are not known. On the other hand, the problem of proving general global existence theorems for the Einstein equations is beyond the reach of the mathematics presently available. To make some progress, it is necessary to concentrate on simplified models. The most common simplifications are to look at solutions with various types of symmetry and solutions for small data. These two approaches are reviewed in sections 3 and 5 respectively. A different approach is to prove the existence of solutions with a prescribed singularity structure. This is discussed in section 6. Section 7 collects some miscellaneous results which cannot easily be classified. With the motivation that insights about the properties of solutions of the Einstein equations can be obtained from the comparison with Newtonian theory and special relativity, relevant results from those areas are presented in section 4.

The area of research reviewed in the following relies heavily on the theory of differential equations, particularly that of hyperbolic partial differential equations. For the benefit of readers with little background in differential equations, some general references which the author has found to be useful will be listed. A thorough introduction to ordinary differential equations is given in [89]. A lot of intuition for ordinary differential equations can be obtained from[94]. The article[10] is full of information, in rather compressed form. A classic introductory text on partial differential equations, where hyperbolic equations are well represented, is [100]. Useful texts on hyperbolic equations, some of which explicitly deal with the Einstein equations, are [160, 104, 126, 117, 158, 101, 68]

An important aspect of existence theorems in general relativity which one should be aware of is their relation to the cosmic censorship hypothesis. This

point of view was introduced in an influential paper by Moncrief and Eardley[121]. An extended discussion of the idea can be found in[59]

2 Local existence

2.1 The constraints

The unknowns in the constraint equations are the initial data for the Einstein equations. These consist of a three-dimensional manifold S, a Riemannian metric h_{ab} and a symmetric tensor k_{ab} on S, and initial data for any matter fields present. The equations are:

$$R - k_{ab}k^{ab} + (h^{ab}k_{ab})^2 = 16\pi\rho \tag{1}$$

$$\nabla^a k_{ab} - \nabla_b(h^{ac}k_{ac}) = 8\pi j_b \tag{2}$$

Here R is the scalar curvature of the metric h_{ab} and ρ and j_a are projections of the energy-momentum tensor. Assuming matter fields which satisfy the dominant energy condition implies that $\rho \geq (j_a j^a)^{1/2}$. This means that the trivial procedure of making an arbitrary choice of h_{ab} and k_{ab} and defining ρ and j_a by equations (1) and (2) is of no use for producing physically interesting solutions.

The usual method for solving the equations (1) and (2) is the conformal method[41]. In this method parts of the data (the so-called free data) are chosen, and the constraints imply four elliptic equations for the remaining parts. The case which has been studied most is the constant mean curvature (CMC) case, where $\mathrm{tr}k = h^{ab}k_{ab}$ is constant. In that case there is an important simplification. Three of the elliptic equations, which form a linear system, decouple from the remaining one. This last equation, which is nonlinear, but scalar, is called the Lichnerowicz equation. The heart of the existence theory for the constraints in the CMC case is the theory of the Lichnerowicz equation.

Solving an elliptic equation is a non-local problem and so boundary conditions or asymptotic conditions are important. For the constraints the cases most frequently considered in the literature are that where S is compact (so that no boundary conditions are needed) and that where the free data satisfy some asymptotic flatness conditions. In the CMC case the problem is well understood for both kinds of boundary conditions[32, 55, 95]. The other case which has been studied in detail is that of hyperboloidal data [2]. The kind of theorem which is obtained is that sufficiently differentiable free data, in some cases required to satisfy some global restrictions, can be completed in a unique way to a solution of the constraints.

In the non-CMC case our understanding is much more limited although some results have been obtained in recent years (see [98, 40] and references therein.) It is an important open problem to extend these so that an overview is obtained comparable to that available in the CMC case. Progress on this could also lead to a better understanding of the question, when a spacetime which admits a

compact, or asymptotically flat, Cauchy surface also admits one of constant mean curvature. Up to now there are only isolated examples which exhibit obstructions to the existence of CMC hypersurfaces[13].

It would be interesting to know whether there is a useful concept of the most general physically reasonable solutions of the constraints representing regular initial configurations. Data of this kind should not themselves contain singularities. Thus it seems reasonable to suppose at least that the metric h_{ab} is complete and that the length of k_{ab}, as measured using h_{ab}, is bounded. Does the existence of solutions of the constraints imply a restriction on the topology of S or on the asymptotic geometry of the data? This question is largely open, and it seems that information is available only in the compact and asymptotically flat cases. In the case of compact S, where there is no asymptotic regime, there is known to be no topological restriction. In the asymptotically flat case there is also no topological restriction implied by the constraints beyond that implied by the condition of asymptotic flatness itself[165] This shows in particular that any manifold which is obtained by deleting a point from a compact manifold admits a solution of the constraints satisfying the minimal conditions demanded above. A starting point for going beyond this could be the study of data which are asymptotically homogeneous. For instance, the Schwarzschild solution contains interesting CMC hypersurfaces which are asymptotic to the product of a 2-sphere with the real line. More general data of this kind could be useful for the study of the dynamics of black hole interiors[143]

To sum up, the conformal approach to solving the constraints, which is the standard one up to now, is well understood in the compact, asymptotically flat and hyperboloidal cases under the constant mean curvature assumption, and only in these cases. For some other approaches see [14], [15] and [168].

2.2 The vacuum evolution equations

The main aspects of the local in time existence theory for the Einstein equations can be illustrated by restricting to smooth (i.e. infinitely differentiable) data for the vacuum Einstein equations. The generalizations to less smooth data and matter fields are discussed in sections 2.3 and 2.4 respectively. In the vacuum case the data are h_{ab} and k_{ab} on a three-dimensional manifold S, as discussed in section 2.1. A solution corresponding to these data is given by a four-dimensional manifold M, a Lorentz metric $g_{\alpha\beta}$ on M and an embedding of S in M. Here $g_{\alpha\beta}$ is supposed to be a solution of the vacuum Einstein equations while h_{ab} and k_{ab} are the induced metric and second fundamental form of the embedding, respectively.

The basic local existence theorem says that, given smooth data for the vacuum Einstein equations, there exists a smooth solution of the equations which gives rise to these data[41]. Moreover, it can be assumed that the image of S under the given embedding is a Cauchy surface for the metric $g_{\alpha\beta}$. The latter fact may be expressed loosely, identifying S with its image, by the statement

that S is a Cauchy surface. A solution of the Einstein equations with given initial data having S as a Cauchy surface is called a Cauchy development of those data. The existence theorem is local because it says nothing about the size of the solution obtained. A Cauchy development of given data has many open subsets which are also Cauchy developments of that data.

It is intuitively clear what it means for one Cauchy development to be an extension of another. The extension is called proper if it is strictly larger than the other development. A Cauchy development which has no proper extension is called maximal. The standard global uniqueness theorem for the Einstein equations uses the notion of the maximal development. It is due to Choquet-Bruhat and Geroch[39]. It says that the maximal development of any Cauchy data is unique up to a diffeomorphism which fixes the initial hypersurface. It is also possible to make a statement of Cauchy stability which says that, in an appropriate sense, the solution depends continuously on the initial data. Details on this can be found in [41].

A somewhat stronger form of the local existence theorem is to say that the solution exists on a uniform time interval in all of space. The meaning of this is not a priori clear, due to the lack of a preferred time coordinate in general relativity. The following is a formulation which is independent of coordinates. Let p be a point of S. The temporal extent $T(p)$ of a development of data on S is the supremum of the length of all causal curves in the development passing through p. In this way a development defines a function T on S. The development can be regarded as a solution which exists on a uniform time interval if T is bounded below by a strictly positive constant. For compact S this is a straightforward consequence of Cauchy stability. In the case of asymptotically flat data it is less trivial. In the case of the vacuum Einstein equations it is true, and in fact the function T grows at least linearly at infinity [55].

When proving the above local existence and global uniqueness theorems it is necessary to use some coordinate or gauge conditions. At least no explicitly diffeomorphism-invariant proofs have been found up to now. Introducing these extra elements leads to a system of reduced equations, whose solutions are determined uniquely by initial data in the strict sense, and not just uniquely up to diffeomorphisms. When a solution of the reduced equations has been obtained, it must be checked that it is a solution of the original equations. This means checking that the constraints and gauge conditions propagate. There are many methods for reducing the equations. An overview of the possibilities may be found in [72]

2.3 Questions of differentiability

Solving the Cauchy problem for a system of partial differential equations involves specifying a set of initial data to be considered, and determining the differentiability properties of solutions. Thus two regularity properties are in-

volved - the differentiability of the allowed data, and that of the corresponding solutions. Normally it is stated that for all data with a given regularity, solutions with a certain type of regularity are obtained. For instance in the section 2.2 we chose both types of regularity to be 'infinitely differentiable'. The correspondence between the regularity of data and that of solutions is not a matter of free choice. It is determined by the equations themselves, and in general the possibilities are severely limited. A similar issue arises in the context of the Einstein constraints, where there is a correspondence between the regularity of free data and full data.

The kinds of regularity properties which can be dealt with in the Cauchy problem depends of course on the mathematical techniques available. When solving the Cauchy problem for the Einstein equations it is necessary to deal at least with nonlinear systems of hyperbolic equations. (There may be other types of equations involved, but they will be ignored here.) For general nonlinear systems of hyperbolic equations there is essentially only one technique known, the method of energy estimates. This method is closely connected with Sobolev spaces, which will now be discussed briefly.

Let u be a real-valued function on $\mathbf{R}^n$. Let:

$$\|u\|_s = (\sum_{i=0}^{s} \int |D^i u|^2(x)dx)^{1/2}$$

The space of functions for which this quantity is finite is the Sobolev space $H^s(\mathbf{R}^n)$. Here $|D^i u|^2$ denotes the sum of the squares of all partial derivatives of u of order i. Thus the Sobolev space H^s is the space of functions, all of whose partial derivatives up to order s are square integrable. Similar spaces can be defined for vector valued functions by taking a sum of contributions from the separate components in the integral. It is also possible to define Sobolev spaces on any Riemannian manifold, using covariant derivatives. General information on this can be found in [11]. Consider now a solution u of the wave equation in Minkowski space. Let $u(t)$ be the restriction of this function to a time slice. Then it is easy to compute that, provided u is smooth and $u(t)$ has compact support for each t, the quantity $\|Du(t)\|_s^2 + \|\partial_t u(t)\|_s^2$ is time independent for each s. For $s = 0$ this is just the energy of a solution of the wave equation. For a general nonlinear hyperbolic system, the Sobolev norms are no longer time-independent. The constancy in time is replaced by certain inequalities. Due to the similarity to the energy for the wave equation, these are called energy estimates. They constitute the foundation of the theory of hyperbolic equations. It is because of these estimates that Sobolev spaces are natural spaces of initial data in the Cauchy problem for hyperbolic equations. The energy estimates ensure that a solution evolving from data belonging to a given Sobolev space on one spacelike hypersurface will induce data belonging to the same Sobolev space on later spacelike hypersurfaces. In other words, the property of belonging to a Sobolev space is propagated by the equations. Due to the locality properties

of hyperbolic equations (existence of a finite domain of dependence), it is useful to introduce the spaces $H^s_{\rm loc}$ which are defined by the condition that whenever the domain of integration is restricted to a compact set the integral defining the space H^s is finite.

In the end the solution of the Cauchy problem should be a function which is differentiable enough in order that all derivatives which occur in the equation exist in the usual (pointwise) sense. A square integrable function is in general defined only almost everywhere and the derivatives in the above formula must be interpreted as distributional derivatives. For this reason a connection between Sobolev spaces and functions whose derivatives exist pointwise is required. This is provided by the Sobolev embedding theorem. This says that if a function u on $\mathbf{R}^n$ belongs to the Sobolev space $H^s_{\rm loc}$ and if $k < s - n/2$ then there is a k times continuously differentiable function which agrees with u except on a set of measure zero.

In the existence and uniqueness theorems stated in section 2.2, the assumptions on the initial data for the vacuum Einstein equations can be weakened to say that h_{ab} should belong to $H^s_{\rm loc}$ and k_{ab} to $H^{s-1}_{\rm loc}$. Then, provided s is large enough, a solution is obtained which belongs to $H^s_{\rm loc}$. In fact its restriction to any spacelike hypersurface also belongs to $H^s_{\rm loc}$, a property which is a priori stronger. The details of how large s must be would be out of place here, since they involve examining the detailed structure of the energy estimates. However there is a simple rule for computing the required value of s. The value of s needed to obtain an existence theorem for the Einstein equations is that for which the Sobolev embedding theorem, applied to spatial slices, just ensures that the metric is continuously differentiable. Thus the requirement is that $s > n/2 + 1 = 5/2$, since $n = 3$. It follows that the smallest possible integer s is three. Strangely enough, uniqueness up to diffeomorphisms is only known to hold for $s \geq 4$. The reason is that in proving the uniqueness theorem a diffeomorphism must be carried out, which need not be smooth. This apparently leads to a loss of one derivative. It would be desirable to show that uniqueness holds for $s = 3$ and to close this gap, which has existed for many years. There exists a definition of Sobolev spaces for an arbitrary real number s, and hyperbolic equations can also be solved in the spaces with s not an integer [159]. Presumably these techniques could be applied to prove local existence for the Einstein equations with s any real number greater than 5/2. However this has apparently not been done explicitly in the literature.

Consider now C^∞ initial data. Corresponding to these data there is a development of class H^s for each s. It could conceivably be the case that the size of these developments shrinks with increasing s. In that case their intersection might contain no open neighbourhood of the initial hypersuface, and no smooth development would be obtained. Fortunately it is known that the H^s developments cannot shrink with increasing s, and so the existence of a C^∞ solution is obtained for C^∞ data. It appears that the H^s spaces with $s > 5/2$ are the only spaces containing the space of smooth functions for which it has been proved

that the Einstein equations are locally solvable.

What is the motivation for considering regularity conditions other than the apparently very natural C^∞ condition? One motivation concerns matter fields and will be discussed in section 2.4. Another is the idea that assuming the existence of many derivatives which have no direct physical significance seems like an admission that the problem has not been fully understood. A further reason for considering low regularity solutions is connected to the possibility of extending a local existence result to a global one. If the proof of a local existence theorem is examined closely it is generally possible to give a continuation criterion. This is a statement that if a solution on a finite time interval is such that a certain quantity constructed from the solution is bounded on that interval, then the solution can be extended to a longer time interval. (In applying this to the Einstein equations we need to worry about introducing an appropriate time coordinate.) If it can be shown that the relevant quantity is bounded on any finite time interval where a solution exists, then global existence follows. It suffices to consider the maximal interval on which a solution is defined, and obtain a contradiction if that interval is finite. This description is a little vague, but contains the essence of a type of argument which is often used in global existence proofs. The problem in putting it into practise is that often the quantity whose boundedness has to be checked contains many derivatives, and is therefore difficult to control. If the continuation criterion can be improved by reducing the number of derivatives required, then this can be a significant step towards a global result. Reducing the number of derivatives in the continuation criterion is closely related to reducing the number of derivatives of the data required for a local existence proof.

A striking example is provided by the work of Klainerman and Machedon [109] on the Yang-Mills equations in Minkowski space. Global existence in this case was first proved by Eardley and Moncrief [66], assuming initial data of sufficiently high differentiability. Klainerman and Machedon gave a new proof of this which, though technically complicated, is based on a conceptually simple idea. They prove a local existence theorem for data of finite energy. Since energy is conserved this immediately proves global existence. In this case finite energy corresponds to the Sobolev space H^1 for the gauge potential. Of course a result of this kind cannot be expected for the Einstein equations, since spacetime singularities do sometimes develop from regular initial data. However, some weaker analogue of the result could exist.

2.4 Matter fields

Analogues of the results for the vacuum Einstein equations given in section 2.2 are known for the Einstein equations coupled to many types of matter model. These include perfect fluids, elasticity theory, kinetic theory, scalar fields, Maxwell fields, Yang-Mills fields and combinations of these. An important restriction is that the general results for perfect fluids and elasticity apply

only to situations where the energy density is uniformly bounded away from zero on the region of interest. In particular they do not apply to cases representing material bodies surrounded by vacuum. In cases where the energy density, while everywhere positive, tends to zero at infinity, a local solution is known to exist, but it is not clear whether a local existence theorem can be obtained which is uniform in time. In cases where the fluid has a sharp boundary, ignoring the boundary leads to solutions of the Einstein-Euler equations with low differentiability (cf. section 2.3), while taking it into account explicitly leads to a free boundary problem. This will be discussed in more detail in section 2.5. In the case of kinetic or field theoretic matter models it makes no difference whether the energy density vanishes somewhere or not.

2.5 Free boundary problems

In applying general relativity one would like to have solutions of the Einstein-matter equations modelling material bodies. As will be discussed in section 3.1 there are solutions available for describing equilibrium situations. However dynamical situations require solving a free boundary problem if the body is to be made of fluid or an elastic solid. We will now discuss the few results which are known on this subject. For a spherically symmetric self-gravitating fluid body in general relativity a local in time existence theorem was proved in [108]. This concerned the case where the density of the fluid at the boundary is non-zero. In [139] a local existence theorem was proved for certain equations of state with vanishing boundary density. These solutions need not have any symmetry but they are very special in other ways. In particular they do not include small perturbations of the stationary solutions discussed in section 3.1. There is no general result on this problem up to now.

Remarkably, the free boundary problem for a fluid body is also poorly understood in classical physics. There is a result for a viscous fluid [153] but in the case of a perfect fluid the problem was wide open until very recently. Now a major step forward has been taken by Wu[167], who obtained a result for a fluid which is incompressible and irrotational. There is a good physical reason why local existence for a fluid with a free boundary might fail. This is the Rayleigh-Taylor instability which involves perturbations of fluid interfaces which grow with unbounded exponential rates. (Cf. the discussion in [18].) It turns out that in the case considered by Wu this instability does not cause problems and there is no reason to expect that a self-gravitating compressible fluid with rotation in general relativity with a free boundary cannot also be described by a well-posed free boundary value problem.

One of the problems in tackling the initial value problem for a dynamical fluid body is that the boundary is moving. It would be very convenient to use Lagrangian coordinates, since in those coordinates the boundary is fixed. Unfortunately, it is not at all obvious that the Euler equations in Lagrangian coordinates have a well-posed initial value problem, even in the absence of a

boundary. It was, however, recently shown by Friedrich[73] that it is possible to treat the Cauchy problem for fluids in general relativity in Lagrangian coordinates.

3 Global symmetric solutions

3.1 Stationary solutions

Many of the results on global solutions of the Einstein equations involve considering classes of spacetimes with Killing vectors. A particularly simple case is that of a timelike Killing vector, i.e. the case of stationary spacetimes. In the vacuum case there are very few solutions satisfying physically reasonable boundary conditions. This is related to no hair theorems for black holes and lies outside the scope of this review. More information on the topic can be found in the book of Heusler [93] and in his Living Review[92]. The case of phenomenological matter models has been reviewed in [148]. The account given there will be updated in the following.

The area of stationary solutions of the Einstein equations coupled to field theoretic matter models has been active in recent years as a consequence of the discovery by Bartnik and McKinnon[16] of a discrete family of regular static spherically symmetric solutions of the Einstein-Yang-Mills equations with gauge group $SU(2)$. The equations to be solved are ordinary differential equations and in [16] they were solved numerically by a shooting method. The first existence proof for a solution of this kind is due to Smoller, Wasserman, Yau and McLeod [157] and involves an arduous qualitative analysis of the differential equations. The work on the Bartnik-McKinnon solutions, including the existence theorems, has been extended in many directions. Recently static solutions of the Einstein-Yang-Mills equations which are not spherically symmetric were discovered numerically [111]. It is a challenge to prove the existence of solutions of this kind. Now the ordinary differential equations of the previously known case are replaced by elliptic equations. Moreover, the solutions appear to still be discrete, so that a simple perturbation argument starting from the spherical case does not seem feasible. In another development it was shown that a linearized analysis indicates the existence of stationary non-static solutions [30]. It would be desirable to study the question of linearization stability in this case, which, if the answer were favourable, would give an existence proof for solutions of this kind.

Now we return to phenomenological matter models, starting with the case of spherically symmetric static solutions. Basic existence theorems for this case have been proved for perfect fluids[149], collisionless matter [134], [129] and elastic bodies [124]. The last of these is the solution to an open problem posed in [148]. All these theorems demonstrate the existence of solutions which are everywhere smooth and exist globally as functions of area radius for a general

class of constitutive relations. The physically significant question of the finiteness of the mass of these configurations was only answered in these papers under restricted circumstances. For instance, in the case of perfect fluids and collisionless matter, solutions were constructed by perturbing about the Newtonian case. Solutions for an elastic body were obtained by perturbing about the case of isotropic pressure, which is equivalent to a fluid. Further progress on the question of the finiteness of the mass of the solutions was made in the case of a fluid by Makino[118], who gave a rather general criterion on the equation of state ensuring the finiteness of the radius. Makino's criterion was generalized to kinetic theory in [132]. This resulted in existence proofs for various models which have been considered in galactic dynamics and which had previously been constructed numerically. (Cf. [24], [154] for an account of these models in the non-relativistic and relativistic cases respectively.)

In the case of self-gravitating Newtonian spherically symmetric configurations of collisionless matter, it can be proved that the phase space density of particles depends only on the energy of the particle and the modulus of its angular momentum[17]. This is known as Jeans' theorem. It was already shown in [129] that the naive generalization of this to the general relativistic case does not hold if a black hole is present. Recently counterexamples to the generalization of Jeans' theorem to the relativistic case which are not dependent on a black hole were constructed by Schaeffer[152]. It remains to be seen whether there might be a natural modification of the formulation which would lead to a true statement.

For a perfect fluid there are results stating that a static solution is necessarily spherically symmetric[114]. They still require a restriction on the equation of state which it would be desirable to remove. A similar result is not to be expected in the case of other matter models, although as yet no examples of non-spherical static solutions are available. In the Newtonian case examples have been constructed by Rein [127]. (In that case static solutions are defined to be those where the particle current vanishes.) For a fluid there is an existence theorem for solutions which are stationary but not static (models for rotating stars)[91]. At present there are no corresponding theorems for collisionless matter or elastic bodies. In [127] stationary, non-static configurations of collisionless matter were constructed in the Newtonian case.

For some remarks on the question of stability see section 4.1.

3.2 Spatially homogeneous solutions

A solution of the Einstein equations is called spatially homogeneous if there exists a group of symmetries with three-dimensional spacelike orbits. In this case there are at least three linearly independent spacelike Killing vector fields. For most matter models the field equations reduce to ordinary differential equations. (Kinetic matter leads to an integro-differential equation.) The most important results in this area have been reviewed in a recent book edited by Wainwright

and Ellis[162]. See, in particular, part two of the book. There remain a host of interesting and accessible open questions. The spatially homogeneous solutions have the advantage that it is not necessary to stop at just existence theorems; information on the global qualitative behaviour of solutions can also be obtained.

An important question which has been open for a long time concerns the mixmaster model, as discussed in [146]. This is a class of spatially homogeneous solutions of the vacuum Einstein equations which are invariant under the group $SU(2)$. A special subclass of these $SU(2)$-invariant solutions, the (parameter-dependent) Taub-NUT solution, is known explicitly in terms of elementary functions. The Taub-NUT solution has a simple initial singularity which is in fact a Cauchy horizon. All other vacuum solutions admitting a transitive action of $SU(2)$ on spacelike hypersurfaces (Bianchi type IX solutions) will be called generic in the present discussion. These generic Bianchi IX solutions (which might be said to constitute the mixmaster solution proper) have been believed for a long time to have singularities which are oscillatory in nature where some curvature invariant blows up. This belief was based on a combination of heuristic considerations and numerical calculations. Although these together do make a persuasive case for the accepted picture, until very recently there were no mathematical proofs of the these features of the mixmaster model available. This has now changed. First, a proof of curvature blow-up and oscillatory behaviour for a simpler model (a solution of the Einstein-Maxwell equations) which shares many qualitative features with the mixmaster model was obtained by Weaver[164]. In the much more difficult case of the mixmaster model itself corresponding results were obtained by Ringström[151]. Forthcoming work of Ringström extends this analysis to prove the correctness of other properties of the mixmaster model suggested by heuristic and numerical work.

Ringström's analysis of the mixmaster model is potentially of great significance for the mathematical understanding of singularities of the Einstein equations in general. Thus its significance goes far beyond the spatially homogeneous case. According to extensive investigations of Belinskii, Khalatnikov and Lifshitz (see [112], [19], [20] and references therein) the mixmaster model should provide an approximate description for the general behaviour of solutions of the Einstein equations near singularities. This should apply to many matter models as well as to the vacuum equations. The work of Belinskii, Khalatnikov and Lifshitz (BKL) is hard to understand and it is particularly difficult to find a precise mathematical formulation of their conclusions. This has caused many people to remain sceptical about the validity of the BKL picture. Nevertheless, it seems that nothing has ever been found which indicates any significant flaws in the final version. As long as the mixmaster model itself was not understood this represented a fundamental obstacle to progress on understanding the BKL picture mathematically. The removal of this barrier opens up an avenue to progress on this issue.

Some recent and qualitatively new results concerning the asymptotic behaviour of spatially homogeneous solutions of the Einstein-matter equations,

both close to the initial singularity and in a phase of unlimited expansion, (and with various matter models) can be found in [150] and [163]. These show in particular that the dynamics can depend sensitively on the form of matter chosen. (Note that these results are consistent with the BKL picture.)

3.3 Spherically symmetric solutions

The most extensive results on global inhomogeneous solutions of the Einstein equations obtained up to now concern spherically symmetric solutions of the Einstein equations coupled to a massless scalar field with asymptotically flat initial data. In a series of papers Christodoulou [43, 42, 45, 44, 46, 47, 48, 52] has proved a variety of deep results on the global structure of these solutions. Particularly notable are his proofs that naked singularities can develop from regular initial data [48] and that this phenomenon is unstable with respect to perturbations of the data [52]. In related work Christodoulou [49, 50, 51] has studied global spherically symmetric solutions of the Einstein equations coupled to a fluid with a special equation of state (the so-called two-phase model).

The rigorous investigation of the spherically symmetric collapse of collisionless matter in general relativity was initiated by Rein and the author [133], who showed that the evolution of small initial data leads to geodesically complete spacetimes where the density and curvature fall off at large times. Later it was shown[136] that independent of the size of the initial data the first singularity, if there is one at all, must occur at the centre of symmetry. This result uses a time coordinate of Schwarzschild type; an analogous result for a maximal time coordinate was proved in [147]. The question of what happens for general large initial data could not yet be answered by analytical techniques. In [137] numerical methods were applied in order to try to make some progress in this direction. The results are discussed in the next paragraph.

Despite the range and diversity of the results obtained by Christodoulou on the spherical collapse of a scalar field, they do not encompass some of the most interesting phenomena which have been observed numerically. These are related to the issue of critical collapse. For sufficiently small data the field disperses. For sufficiently large data a black hole is formed. The question is what happens in between. This can be investigated by examining a one-parameter family of initial data interpolating between the two cases. It was found by Choptuik[38] that there is a critical value of the parameter below which dispersion takes place and above which a black hole is formed and that the mass of the black hole approaches zero as the critical parameter value is approached. This gave rise to a large literature where the spherical collapse of different kinds of matter was computed numerically and various qualitative features were determined. For a review of this see [85]. In the calculations of [137] for collisionless matter it was found that in the situations considered the black hole mass tended to a strictly positive limit as the critical parameter was approached from above. There are no rigorous mathematical results available on the issue of a mass gap

for either a scalar field or collisionless matter and it is an outstanding challenge for mathematical relativists to change this situation.

Another aspect of Choptuik's results is the occurrence of a discretely self-similar solution. It would seem hard to prove the existence of a solution of this kind analytically. For other types of matter, such as a perfect fluid with linear equation of state, the critical solution is continuously self-similar and this looks more tractable. The problem reduces to solving a system of singular ordinary differential equations subject to certain boundary conditions. A problem of this type was solved in [48], but the solutions produced there, which are continuously self-similar, cannot include the Choptuik critical solution. In the case of a perfect fluid the existence of the critical solution seems to be a problem which could possibly be solved in the near future. A good starting point for this is the work of Goliath, Nilsson and Uggla[82], [83]. These authors gave a formulation of the problem in terms of dynamical systems and were able to determine certain qualitative features of the solutions.

3.4 Cylindrically symmetric solutions

Solutions of the Einstein equations with cylindrical symmetry which are asymptotically flat in all directions allowed by the symmetry represent an interesting variation on asymptotic flatness. Since black holes are apparently incompatible with this symmetry, one may hope to prove geodesic completeness of solutions under appropriate assumptions. (It would be interesting to have a theorem making the statement about black holes precise.) A proof of geodesic completeness has been achieved for the Einstein vacuum equations and for the source-free Einstein-Maxwell equations in [22], building on global existence theorems for wave maps[57, 56]. For a quite different point of view on this question involving integrable systems see [166]. A recent preprint of Hauser and Ernst[90] also appears to be related to this question. However, due to the great length of this text and its reliance on many concepts unfamiliar to this author, no further useful comments on the subject can be made here.

3.5 Spatially compact solutions

In the context of spatially compact spacetimes it is first necessary to ask what kind of global statements are to be expected. In a situation where the model expands indefinitely it is natural to pose the question whether the spacetime is causally geodesically complete towards the future. In a situation where the model develops a singularity either in the past or in the future one can ask what the qualitative nature of the singularity is. It is very difficult to prove results of this kind. As a first step one may prove a global existence theorem in a well-chosen time coordinate. In other words, a time coordinate is chosen which is geometrically defined and which, under ideal circumstances, will take all values in a certain interval (t_-, t_+). The aim is then to show that, in the

maximal Cauchy development of data belonging to a certain class, a time coordinate of the given type exists and exhausts the expected interval. The first result of this kind for inhomogeneous spacetimes was proved by Moncrief in [119]. This result concerned Gowdy spacetimes. These are vacuum spacetimes with two commuting Killing vectors acting on compact orbits. The area of the orbits defines a natural time coordinate. Moncrief showed that in the maximal Cauchy development of data given on a hypersurface of constant time, this time coordinate takes on the maximal possible range, namely $(0, \infty)$. This result was extended to more general vacuum spacetimes with two Killing vectors in [21]. Andréasson[4] extended it in another direction to the case of collisionless matter in a spacetime with Gowdy symmetry.

Another attractive time coordinate is constant mean curvature (CMC) time. For a general discussion of this see [143]. A global existence theorem in this time for spacetimes with two Killing vectors and certain matter models (collisionless matter, wave maps) was proved in [145]. That the choice of matter model is important for this result was demonstrated by a global non-existence result for dust in [144]. As shown in [99], this leads to the examples of spacetimes which are not covered by a CMC slicing. Related results have been obtained for spherical and hyperbolic symmetry [142, 31]. The results of [145] and [4] have many analogous features and it would be desirable to establish connections between them, since this might lead to results stronger than those obtained by either of the techniques individually.

Once global existence has been proved for a preferred time coordinate, the next step is to investigate the asymptotic behaviour of the solution as $t \to t_{\pm}$. There are few cases in which this has been done successfully. Notable examples are Gowdy spacetimes [58, 97, 61] and solutions of the Einstein-Vlasov system with spherical and plane symmetry[130]. Progress in constructing spacetimes with prescribed singularities will be described in section 6. In the future this could lead in some cases to the determination of the asymptotic behaviour of large classes of spacetimes as the singularity is approached.

4 Newtonian theory and special relativity

To put the global results discussed in this article into context it is helpful to compare with Newtonian theory and special relativity. Some of the theorems which have been proved in those contexts and which can offer insight into questions in general relativity will now be reviewed. It should be noted that even in these simpler contexts open questions abound.

4.1 Hydrodynamics

Solutions of the classical (compressible) Euler equations typically develop singularities, i.e. discontinuities of the basic fluid variables, in finite time[155].

Some of the results of [155] were recently generalized to the case of a relativistic fluid[88]. The proofs of the development of singularities are by contradiction and so do not give information about what happens when the smooth solution breaks down. One of the things which can happen is the formation of shock waves and it is known that at least in certain cases solutions can be extended in a physically meaningful way beyond the time of shock formation. The extended solutions only satisfy the equations in the weak sense. For the classical Euler equations there is a well-known theorem on global existence of classical solutions in one space dimension which goes back to [81]. This has been generalized to the relativistic case. Smoller and Temple treated the case of an isentropic fluid with linear equation of state[156] while Chen analysed the cases of polytropic equations of state[36] and flows with variable entropy [37]. This means that there is now an understanding of this question in the relativistic case similar to that available in the classical case.

In space dimensions higher than one there are no general global existence theorems. For a long time there were also no uniqueness theorems for weak solutions even in one dimension. It should be emphasized that weak solutions can easily be shown to be non-unique unless they are required to satisfy additional restrictions such as entropy conditions. A reasonable aim is to find a class of weak solutions in which existence and uniqueness hold. In the one-dimensional case this has recently been achieved by Bressan and collaborators (see [28], [29] and references therein).

It would be desirable to know more about which quantities must blow up when a singularity forms in higher dimensions. A partial answer was obtained for classical hydrodynamics by Chemin[35]. The possibility of generalizing this to relativistic and self-gravitating fluids was studied by Brauer[26]. There is one situation in which a smooth solution of the classical Euler equations is known to exist for all time. This is when the initial data are small and the fluid initially flowing uniformly outwards. A theorem of this type has been proved by Grassin[84]. There is also a global existence result due to Guo[86] for an irrotational charged fluid in Newtonian physics, where the repulsive effect of the charge can suppress the formation of singularities.

A question of great practical interest for physics is that of the stability of equilibrium stellar models. Since, as has already been pointed out, we know so little about the global time evolution for a self-gravitating fluid ball, even in the Newtonian case, it is not possible to say anything rigorous about nonlinear stability at the present time. We can, however, make some statements about linear stability. The linear stability of a large class of static spherically symmetric solutions of the Einstein-Euler equations within the class of spherically symmetric perturbations has been proved by Makino[118]. (Cf. also [113] for the Newtonian problem.) The spectral properties of the linearized operator for general (i.e. non-spherically symmetric) perturbations in the Newtonian problem have been studied by Beyer[23]. This could perhaps provide a basis for a stability analysis, but this has not been done.

4.2 Kinetic theory

Collisionless matter is known to admit a global singularity-free evolution in many cases. For self-gravitating collisionless matter, which is described by the Vlasov-Poisson system, there is a general global existence theorem[125], [116]. There is also a version of this which applies to Newtonian cosmology[135]. A more difficult case is that of the Vlasov-Maxwell system, which describes charged collisionless matter. Global existence is not known for general data in three space dimensions but has been shown in two space dimensions[78], [79] and in three dimensions with one symmetry[77] or with almost spherically symmetric data[128].

The nonlinear stability of static solutions of the Vlasov-Poisson system describing Newtonian self-gravitating collisionless matter has been investigated using the energy-Casimir method. For information on this see [87] and its references.

For the classical Boltzmann equation global existence and uniqueness of smooth solutions has been proved for homogeneous initial data and for data which are small or close to equilibrium. For general data with finite energy and entropy global existence of weak solutions (without uniqueness) was proved by DiPerna and Lions[63]. For information on these results and on the classical Boltzmann equation in general see [34], [33]. Despite the non-uniqueness it is possible to show that all solutions tend to equilibrium at late times. This was first proved by Arkeryd[9] by non-standard analysis and then by Lions[115] without those techniques. It should be noted that since the usual conservation laws for classical solutions are not known to hold for the DiPerna-Lions solutions, it is not possible to predict which equilibrium solution a given solution will converge to. In the meantime analogues of several of these results for the classical Boltzmann equation have been proved in the relativistic case. Global existence of weak solutions was proved in [65]. Global existence and convergence to equilibrium for classical solutions starting close to equilibrium was proved in [80]. On the other hand global existence of classical solutions for small initial data is not known. Convergence to equilibrium for weak solutions with general data was proved by Andréasson[3]. There is still no existence and uniqueness theorem in the literature for general spatially homogeneous solutions of the relativistic Boltzmann equation. (A paper claiming to prove existence and uniqueness for solutions of the Einstein-Boltzmann system which are homogeneous and isotropic[122] contains fundamental errors.)

5 Global existence for small data

An alternative to symmetry assumptions is provided by 'small data' results, where solutions are studied which develop from data close to that for known solutions. This leads to some simplification in comparison to the general problem,

but with present techniques it is still very hard to obtain results of this kind.

5.1 Stability of de Sitter space

In [69] Friedrich proved a result on the stability of de Sitter space. This concerns the Einstein vacuum equations with positive cosmological constant. His result is as follows. Consider initial data induced by de Sitter space on a regular Cauchy hypersurface. Then all initial data (vacuum with positive cosmological constant) near enough to these data in a suitable (Sobolev) topology have maximal Cauchy developments which are geodesically complete. In fact the result gives much more detail on the asymptotic behaviour than just this and may be thought of as proving a form of the cosmic no hair conjecture in the vacuum case. (This conjecture says roughly that the de Sitter solution is an attractor for expanding cosmological models with positive cosmological constant.) This result is proved using conformal techniques and, in particular, the regular conformal field equations developed by Friedrich.

There are results obtained using the regular conformal field equations for negative or vanishing cosmological constant [71, 74] but a detailed discussion of their nature would be out of place here. (Cf. however section 7.1.)

5.2 Stability of Minkowski space

The other result on global existence for small data is that of Christodoulou and Klainerman on the stability of Minkowski space[54] The formulation of the result is close to that given in section 5.1 but now de Sitter space is replaced by Minkowski space. Suppose then that initial data are given which are asymptotically flat and sufficiently close to those induced by Minkowski space on a hyperplane. Then Christodoulou and Klainerman prove that the maximal Cauchy development of these data is geodesically complete. They also provide a wealth of detail on the asymptotic behaviour of the solutions. The proof is very long and technical. The central tool is the Bel-Robinson tensor which plays an analogous role for the gravitational field to that played by the energy-momentum tensor for matter fields. Apart from the book of Christodoulou and Klainerman itself some introductory material on geometric and analytic aspects of the proof can be found in [25] and [53] respectively.

In the original version of the theorem initial data had to be prescribed on all of R^3. A generalization described in [110] concerns the case where data need only be prescribed on the complement of a compact set in R^3. This means that statements can be obtained for any asymptotically flat spacetime where the initial matter distribution has compact support, provided attention is confined to a suitable neighbourhood of infinity. The proof of the new version uses a double null foliation instead of the foliation by spacelike hypersurfaces previously used and leads to certain conceptual simplifications.

5.3 Stability of the Milne model

The interior of the light cone in Minkowski space foliated by the spacelike hypersurfaces of constant Lorentzian distance from the origin can be thought of as a vacuum cosmological model, sometimes known as the Milne model. By means of a suitable discrete subgroup of the Lorentz group it can be compactified to give a spatially compact cosmological model. With a slight abuse of terminology the latter spacetime will also be referred to here as the Milne model. A proof of the stability of the latter model by Andersson and Moncrief has been announced in [1]. The result is that, given data for the Milne model on a manifold obtained by compactifying a hyperboloid in Minkowski space, the maximal Cauchy developments of nearby data are geodesically complete in the future. Moreover the Milne model is asymptotically stable in the sense that any other solution in this class converges towards the Milne model in terms of suitable dimensionless variables.

The techniques used by Andersson and Moncrief are similar to those used by Christodoulou and Klainerman. In particular, the Bel-Robinson tensor is crucial. However their situation is much simpler than that of Christodoulou and Klainerman, so that the complexity of the proof is not so great. This has to do with the fact that the fall-off of the fields in the Minkowksi case towards infinity is different in different directions, while it is uniform in the Milne case. Thus it is enough in the latter case to always contract the Bel-Robinson tensor with the same timelike vector when deriving energy estimates. The fact that the proof is simpler opens up a real possibility of generalizations, for instance by adding different matter models.

6 Prescribed singularities

6.1 Isotropic singularities

The existence and uniqueness results discussed in this section are motivated by Penrose's Weyl curvature hypothesis. Penrose suggests that the initial singularity in a cosmological model should be such that the Weyl tensor tends to zero or at least remains bounded. There is some difficulty in capturing this by a geometric condition and it was suggested in [161] that a clearly formulated geometric condition which, on an intuitive level, is closely related to the original condition, is that the conformal structure should remain regular at the singularity. Singularities of this type are known as conformal or isotropic singularities.

Consider now the Einstein equations coupled to a perfect fluid with the radiation equation of state $p = \rho/3$. Then it has been shown [123, 62] that solutions with an isotropic singularity are determined uniquely by certain free data given at the singularity. The data which can be given is, roughly speaking, half as large as in the case of a regular Cauchy hypersurface. The method of proof is to derive an existence and uniqueness theorem for a suitable class of

singular hyperbolic equations. In [7] this was extended to the equation of state $p = (\gamma - 1)\rho$ for any γ satisfying $1 < \gamma \leq 2$.

What happens to this theory when the fluid is replaced by a different matter model? The study of the case of a collisionless gas of massless particles was initiated in [8]. The equations were put into a form similar to that which was so useful in the fluid case and therefore likely to be conducive to proving existence theorems. Then theorems of this kind were proved in the homogeneous special case. These were extended to the general (i.e. inhomogeneous) case in [5]. The picture obtained for collisionless matter is very different from that for a perfect fluid. Much more data can be given freely at the singularity in the collisionless case.

These results mean that the problem of isotropic singularities has largely been solved. There do, however, remain a couple of open questions. What happens if the massless particles are replaced by massive ones? What happens if the matter is described by the Boltzmann equation with non-trivial collision term? Does the result in that case look more like the Vlasov case or more like the Euler case?

6.2 Fuchsian equations

The singular equations which arise in the study of isotropic singularities are closely related to what Kichenassamy[104] calls Fuchsian equations. He has developed a rather general theory of these equations. (See [104], [103], [102], and also the earlier papers [12], [105] and [106].) In [107] this was applied to analytic Gowdy spacetimes to construct a family of vacuum spacetimes depending on the maximum number of free functions (for the given symmetry class) whose singularities can be described in detail. The symmetry assumed in that paper requires the two-surfaces orthogonal to the group orbits to be surface-forming (vanishing twist constants). In [96] a corresponding result was obtained for the class of vacuum spacetimes with polarized $U(1) \times U(1)$ symmetry and non-vanishing twist.

A result of Anguige[6] is of a similar type but there are several significant differences. He considers perfect fluid spacetimes and can handle smooth data rather than only the analytic case. On the other hand he assumes plane symmetry, which is stronger than Gowdy symmetry.

Related work was done earlier in a somewhat simpler context by Moncrief[120] who showed the existence of a large class of analytic vacuum spacetimes with Cauchy horizons.

7 Further results

7.1 Evolution of hyperboloidal data

In section 2.1 hyperboloidal initial data were mentioned. They can be thought of as generalizations of the data induced by Minkowski space on a hyperboloid. In the case of Minkowski space the solution admits a conformal compactification where a conformal boundary, null infinity, can be added to the spacetime. It can be shown that in the case of the maximal development of hyperboloidal data a piece of null infinity can be attached to the spacetime. For small data, i.e. data close to that of a hyperboloid in Minkowski space, this conformal boundary also has completeness properties in the future allowing an additional point i_+ to be attached there. (See [70] and references therein for more details.) Making contact between hyperboloidal data and asymptotically flat initial data is much more difficult and there is as yet no complete picture. (An account of the results obtained up to now is given in [74].) If the relation between hyperboloidal and asymptotically flat initial data could be understood it would give a very different approach to the problem treated by Christodoulou and Klainerman (section 5.2). It might well also give more detailed information on the asymptotic behaviour of the solutions.

7.2 The Newtonian limit

Most textbooks on general relativity discuss the fact that Newtonian gravitational theory is the limit of general relativity as the speed of light tends to infinity. It is a non-trivial task to give a precise mathematical formulation of this statement. Ehlers systematized extensive earlier work on this problem and gave a precise definition of the Newtonian limit of general relativity which encodes those properties which are desirable on physical grounds (see [67].) Once a definition has been given the question remains whether this definition is compatible with the Einstein equations in the sense that there are general families of solutions of the Einstein equations which have a Newtonian limit in the sense of the chosen definition. A theorem of this kind was proved in [141], where the matter content of spacetime was assumed to be a collisionless gas described by the Vlasov equation. (For another suggestion as to how this problem could be approached see [76].) The essential mathematical problem is that of a family of equations depending continuously on a parameter λ which are hyperbolic for $\lambda \neq 0$ and degenerate for $\lambda = 0$. Because of the singular nature of the limit it is by no means clear a priori that there are families of solutions which depend continuously on λ. That there is an abundant supply of families of this kind is the result of [141]. Asking whether there are families which are k times continuously differentiable in their dependence on λ is related to the issue of giving a mathematical justification of post-Newtonian approximations. The approach of [141] has not even been extended to the case $k = 1$ and it would be desirable to do this. Note however that for k too large serious restrictions arise [140]. The latter fact corresponds to the well-known divergent behaviour of higher order

post-Newtonian approximations.

7.3 Newtonian cosmology

Apart from the interest of the Newtonian limit, Newtonian gravitational theory itself may provide interesting lessons for general relativity. This is no less true for existence theorems than for other issues. In this context it is also interesting to consider a slight generalization of Newtonian theory, the Newton-Cartan theory. This allows a nice treatment of cosmological models, which are in conflict with the (sometimes implicit) assumption in Newtonian gravitational theory that only isolated systems are considered. It is also unproblematic to introduce a cosmological constant into the Newton-Cartan theory.

Three global existence theorems have been proved in Newtonian cosmology. The first[27] is an analogue of the cosmic no hair theorem (cf. section 5.1) and concerns models with a positive cosmological constant. It asserts that homogeneous and isotropic models are nonlinearly stable if the matter is described by dust or a polytropic fluid with pressure. Thus it gives information about global existence and asymptotic behaviour for models arising from small (but finite) perturbations of homogeneous and isotropic data. The second and third results concern collisionless matter and the case of vanishing cosmological constant. The second[135] says that data which constitute a periodic (but not necessarily small) perturbation of a homogeneous and isotropic model which expands indefinitely give rise to solutions which exist globally in the future. The third[131] says that the homogeneous and isotropic models in Newtonian cosmology which correspond to a $k = -1$ Friedmann-Robertson-Walker model in general relativity are non-linearly stable.

7.4 The characteristic initial value problem

In the standard Cauchy problem, which has been the basic set-up for all the previous sections, initial data are given on a spacelike hypersurface. However there is also another possibility, where data are given on one or more null hypersurfaces. This is the characteristic initial value problem. It has the advantage over the Cauchy problem that the constraints reduce to ordinary differential equations. One variant is to give initial data on two smooth null hypersurfaces which intersect transversely in a spacelike surface. A local existence theorem for the Einstein equations with an initial configuration of this type was proved in [138]. Another variant is to give data on a light cone. In that case local existence for the Einstein equations has not been proved, although it has been proved for a class of quasilinear hyperbolic equations which includes the reduced Einstein equations in harmonic coordinates[64].

Another existence theorem which does not use the standard Cauchy problem, and which is closely connected to the use of null hypersurfaces, concerns the Robinson-Trautman solutions of the vacuum Einstein equations. In that case

the Einstein equations reduce to a parabolic equation. Global existence for this equation has been proved by Chruściel[60].

7.5 The initial boundary value problem

In most applications of evolution equations in physics (and in other sciences) initial conditions need to be supplemented by boundary conditions. This leads to the consideration of initial boundary value problems. It is not so natural to consider such problems in the case of the Einstein equations since in that case there are no physically motivated boundary conditions. (For instance, we do not know how to build a mirror for gravitational waves.) An exception is the case of fluid boundary discussed in section 2.5.

For the vacuum Einstein equations it is not a priori clear that it is even possible to find a well-posed initial boundary value problem. Thus it is particularly interesting that Friedrich and Nagy[75] have been able to prove the well-posedness of certain initial boundary value problems for the vacuum Einstein equations. Since boundary conditions come up quite naturally when the Einstein equations are solved numerically, due to the need to use a finite grid, the results of [75] are potentially important for numerical relativity. The techniques developed there could also play a key role in the study of the initial value problem for fluid bodies (Cf. section 2.5.)

8 Acknowledgements

I thank Håkan Andréasson and Bernd Brügmann for helpful suggestions.

References

[1] Andersson, L., "The global existence problem in general relativity", (November, 1999), [Online Los Alamos Archive Preprint]: cited on 4 December 1999, http://xxx.lanl.gov/abs/gr-qc/9911032.

[2] Andersson, L., Chruściel, P. T., and Friedrich, H., "On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations.", *Commun. Math. Phys.*, **149**, 587–612, (1992).

[3] Andréasson, H., "Regularity of the gain term and strong L^1 convergence to equilibrium for the relativistic Boltzmann equation.", *SIAM J. Math. Anal.*, **27**, 1386–1405, (1996).

[4] Andréasson, H., "Global foliations of matter spacetimes with Gowdy symmetry", *Commun. Math. Phys.*, **206**, 337–365, (1999).

[5] Anguige, K., "Isotropic cosmological singularities 3: The Cauchy problem for the inhomogeneous conformal Einstein-Vlasov equations", (March, 1999), [Online Los Alamos Archive Preprint]: cited on 21 September 1999, http://xxx.lanl.gov/abs/gr-qc/9903018.

[6] Anguige, K., "A class of plane symmetric perfect-fluid cosmologies with a Kasner-like singularity", (August, 1999), [Online Los Alamos Archive Preprint]: cited on 21 September 1999, http://xxx.lanl.gov/abs/gr-qc/9908072.

[7] Anguige, K., and Tod, K. P., "Isotropic cosmological singularities 1: Polytropic perfect fluid spacetimes.", *Ann. Phys. (NY)*, **276**, 257–293, (1999).

[8] Anguige, K., and Tod, K. P., "Isotropic cosmological singularities 2: The Einstein-Vlasov system.", *Ann. Phys. (NY)*, **276**, 294–320, (1999).

[9] Arkeryd, L., "On the strong L^1 trend to equilibrium for the Boltzmann equation", *Stud. Appl. Math.*, **87**, 283–288, (1992).

[10] Arnold, V. I., and Ilyashenko, Yu. S., "Ordinary differential equations", in Anosov, D. V., and Arnold, V. I., eds., *Dynamical Systems I.*, 1–148, (Springer, Berlin, 1988).

[11] Aubin, T., *Nonlinear Analysis on Manifolds. Monge-Ampère equations*, (Springer, Berlin, 1982).

[12] Baouendi, M. S., and Goulaouic, C., "Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems", *Commun. P. D. E.*, **2**, 1151–1162, (1977).

[13] Bartnik, R., "Remarks on cosmological spacetimes and constant mean curvature hypersurfaces.", *Commun. Math. Phys.*, **117**, 615–624, (1988).

[14] Bartnik, R., "Quasi-spherical metrics and prescribed scalar curvature.", *J. Diff. Geom.*, **37**, 31–71, (1993).

[15] Bartnik, R., and Fodor, G., "On the restricted validity of the thin sandwich conjecture.", *Phys. Rev. D*, **48**, 3596–3599, (1993).

[16] Bartnik, R., and McKinnon, J., "Particlelike solutions of the Einstein-Yang-Mills equations.", *Phys. Rev. Lett.*, **61**, 141–143, (1988).

[17] Batt, J., Faltenbacher, W., and Horst, E., "Stationary spherically symmetric models in stellar dynamics.", *Arch. Rat. Mech. Anal.*, **93**, 159–183, (1986).

[18] Beale, J. T., Hou, T. Y., and Lowengrub, J. S., "Growth rates for the linearized motion of fluid interfaces away from equilibrium", *Commun. Pure Appl. Math.*, **46**, 1269–1301, (1993).

[19] Belinskii, V. A., Khalatnikov, I. M., and Lifshitz, E. M., “Oscillatory approach to a singular point in the relativistic cosmology.”, *Adv. Phys.*, **19**, 525–573, (1970).

[20] Belinskii, V. A., Khalatnikov, I. M., and Lifshitz, E. M., “A general solution of the Einstein equations with a time singularity.”, *Adv. Phys.*, **31**, 639–667, (1982).

[21] Berger, B. K., Chruściel, P. T., Isenberg, J., and Moncrief, V., “Global foliations of vacuum spacetimes with T^2 isometry”, *Ann. Phys. (NY)*, **260**, 117–148, (1997).

[22] Berger, B. K., Chruściel, P. T., and Moncrief, V., “On asymptotically flat spacetimes with G_2-invariant Cauchy surfaces”, *Ann. Phys. (NY)*, **237**, 322–354, (1995).

[23] Beyer, H., “The spectrum of adiabatic stellar oscillations.”, *J. Math. Phys.*, **36**, 4792–4814, (1995).

[24] Binney, J., and Tremaine, S., *Galactic dynamics*, (Princeton University Press, Princeton, 1987).

[25] Bourguignon, J.-P., “Stabilité par déformation non-linéaire de la métrique de Minkowski (d'après D. Christodoulou et S. Klainerman).”, *Astérisque*, **201-203**, 321–358, (1992).

[26] Brauer, U., “Singularitäten in relativistischen Materiemodellen.”, Thesis, University of Potsdam, 1995.

[27] Brauer, U., Rendall, A. D., and Reula, O. A., “The cosmic no-hair theorem and the nonlinear stability of homogeneous Newtonian cosmological models.”, *Class. Quantum Grav.*, **11**, 2283–2296, (1994).

[28] Bressan, A., “The unique limit of the Glimm scheme.”, *Arch. Rat. Mech. Anal.*, **130**, 205–230, (1995).

[29] Bressan, A., and Colombo, R. M., “The semigroup generated by 2×2 conservation laws.”, *Arch. Rat. Mech. Anal.*, **133**, 1–75, (1995).

[30] Brodbeck, O., Heusler, M., Straumann, N., and Volkov, M., “Rotating solitons and non-rotating non-static black holes.”, *Phys. Rev. Lett.*, **79**, 4310–4313, (1997).

[31] Burnett, G. A., and Rendall, A. D., “Existence of maximal hypersurfaces in some spherically symmetric spacetimes.”, *Class. Quantum Grav.*, **13**, 111–123, (1996).

[32] Cantor, M., “A necessary and sufficient condition for York data to specify an asymptotically flat spacetime.”, *J. Math. Phys.*, **20**, 1741–1744, (1979).

[33] Cercignani, C., *The Boltzmann equation and its applications*, (Springer, Berlin, 1994).

[34] Cercignani, C., Illner, R., and Pulvirenti, M., *The mathematical theory of dilute gases*, (Springer, Berlin, 1988).

[35] Chemin, J.-Y., "Remarques sur l'apparition de singularités dans les écoulements Euleriens compressibles", *Commun. Math. Phys.*, **133**, 323–339, (1990).

[36] Chen, J., "Conservation laws for the relativistic p-system", *Comm. Partial Diff. Eq.*, **20**, 1605–1646, (1995).

[37] Chen, J., "Conservation laws for relativistic fluid dynamics", *Arch. Rat. Mech. Anal.*, **139**, 377–398, (1997).

[38] Choptuik, M. W., "Universality and scaling in the gravitational collapse of a scalar field.", *Phys. Rev. Lett.*, **70**, 9–12, (1993).

[39] Choquet-Bruhat, Y., and Geroch, R., "Global aspects of the Cauchy problem in general relativity.", *Commun. Math. Phys.*, **14**, 329–335, (1969).

[40] Choquet-Bruhat, Y., Isenberg, J., and York, J. W., "Einstein constraints on asymptotically Euclidean manifolds", (June, 1999), [Online Los Alamos Archive Preprint]: cited on 20 August 1999, http://xxx.lanl.gov/abs/gr-qc/9906095.

[41] Choquet-Bruhat, Y., and York, J., "The Cauchy problem.", in Held, A., ed., *General relativity and gravitation.*, volume 1, 99–172, (Plenum, New York, 1980).

[42] Christodoulou, D., "Global existence of generalised solutions of the spherically symmetric Einstein-scalar equations in the large.", *Commun. Math. Phys.*, **106**, 587–621, (1986).

[43] Christodoulou, D., "The problem of a self-gravitating scalar field.", *Commun. Math. Phys.*, **105**, 337–361, (1986).

[44] Christodoulou, D., "A mathematical theory of gravitational collapse.", *Commun. Math. Phys.*, **109**, 613–647, (1987).

[45] Christodoulou, D., "The structure and uniqueness of generalised solutions of the spherically symmetric Einstein-scalar equations.", *Commun. Math. Phys.*, **109**, 591–611, (1987).

[46] Christodoulou, D., "The formation of black holes and singularities in spherically symmetric gravitational collapse.", *Commun. Pure Appl. Math.*, **44**, 339–373, (1991).

[47] Christodoulou, D., "Bounded variation solutions of the spherically symmetric Einstein-scalar field equations.", *Commun. Pure Appl. Math.*, **46**, 1131–1220, (1993).

[48] Christodoulou, D., "Examples of naked singularity formation in the gravitational collapse of a scalar field.", *Ann. Math.*, **140**, 607–653, (1994).

[49] Christodoulou, D., "Self-gravitating fluids: a two-phase model.", *Arch. Rat. Mech. Anal.*, **130**, 343–400, (1995).

[50] Christodoulou, D., "Self-gravitating fluids: the continuation and termination of a free phase boundary.", *Arch. Rat. Mech. Anal.*, **133**, 333–398, (1996).

[51] Christodoulou, D., "Self-gravitating fluids: the formation of a free phase boundary in the phase transition from soft to hard.", *Arch. Rat. Mech. Anal.*, **134**, 97–154, (1996).

[52] Christodoulou, D., "The instability of naked singularities in the gravitational collapse of a scalar field.", *Ann. Math.*, **149**, 183–217, (1999).

[53] Christodoulou, D., and Klainerman, S., "Asymptotic properties of linear field equations in Minkowski space.", *Commun. Pure Appl. Math.*, **43**, 137–199, (1990).

[54] Christodoulou, D., and Klainerman, S., *The global nonlinear stability of the Minkowski space.*, (Princeton University Press, Princeton, 1993).

[55] Christodoulou, D., and Ó Murchadha, N., "The boost problem in general relativity.", *Commun. Math. Phys.*, **10**, 271–300, (1981).

[56] Christodoulou, D., and Tahvildar-Zadeh, A. S., "On the asymptotic behaviour of spherically symmetric wave maps.", *Duke Math. J,*, **71**, 31–69, (1993).

[57] Christodoulou, D., and Tahvildar-Zadeh, A. S., "On the regularity of spherically symmetric wave maps.", *Commun. Pure Appl. Math.*, **46**, 1041–1091, (1993).

[58] Chruściel, P. T., "On spacetimes with $U(1) \times U(1)$ symmetric compact Cauchy surfaces.", *Ann. Phys. (NY)*, **202**, 100–150, (1990).

[59] Chruściel, P. T., *On the uniqueness in the large of solutions of Einstein's equations. (Strong cosmic censorship.)*, volume 20 of *Proc. Centre Math. Anal.*, (Australian National University, Canberra, 1991).

[60] Chruściel, P. T., "Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation.", *Commun. Math. Phys.*, **137**, 289–313, (1991).

[61] Chruściel, P. T., Isenberg, J., and Moncrief, V., "Strong cosmic censorship in polarised Gowdy spacetimes.", *Class. Quantum Grav.*, **7**, 1671–1680, (1990).

[62] Claudel, C. M., and Newman, K. P., "The Cauchy problem for quasi-linear hyperbolic evolution problems with a singularity in the time.", *Proc. R. Soc. Lond. A*, **454**, 1073–1107, (1998).

[63] DiPerna, R. J., and Lions, P.-L., "On the Cauchy problem for Boltzmann equations: global existence and weak stability.", *Ann. Math.*, **130**, 321–366, (1989).

[64] Dossa, M., "Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique.", *Ann. Inst. H. Poincaré (Phys. Théor.)*, **66**, 37–107, (1997).

[65] Dudynski, M., and Ekiel-Jezewska, M., "Global existence proof for the relativistic Boltzmann equation.", *J. Stat. Phys.*, **66**, 991–1001, (1992).

[66] Eardley, D., and Moncrief, V., "The global existence of Yang-Mills fields in M^{3+1}.", *Commun. Math. Phys.*, **83**, 171–212, (1982).

[67] Ehlers, J., "The Newtonian limit of general relativity", in Ferrarese, G., ed., *Classical mechanics and relativity: relationship and consistency*, (Bibliopolis, Naples, 1991).

[68] Evans, L. C., *Partial differential equations.*, (American Mathematical Society, Providence, 1998).

[69] Friedrich, H., "Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant.", *J. Geom. Phys.*, **3**, 101–117, (1986).

[70] Friedrich, H., "On the global existence and asymptotic behaviour of solutions to the Einstein-Yang-Mills equations.", *J. Diff. Geom.*, **34**, 275–345, (1991).

[71] Friedrich, H., "Einstein equations and conformal structure: Existence of anti-de Sitter-type spacetimes.", *J. Geom. Phys.*, **17**, 125–184, (1995).

[72] Friedrich, H., "Hyperbolic reductions of Einstein's field equations.", *Class. Quantum Grav.*, **13**, 1451–1469, (1996).

[73] Friedrich, H., "Evolution equations for gravitating ideal fluid bodies in general relativity.", *Phys. Rev. D*, **D57**, 2317–2322, (1998).

[74] Friedrich, H., "Gravitational fields near spacelike and null infinity.", *J. Geom. Phys.*, **24**, 83–172, (1998).

[75] Friedrich, H., and Nagy, G., "The initial boundary value problem for Einstein's vacuum field equations.", *Commun. Math. Phys.*, **201**, 619–655, (1999).

[76] Fritelli, S., and Reula, O., "On the Newtonian limit of general relativity.", *Commun. Math. Phys.*, **166**, 221–235, (1994).

[77] Glassey, R. T., and Schaeffer, J., "The 'two and one half dimensional' relativistic Vlasov-Maxwell system.", *Commun. Math. Phys.*, **185**, 257–284, (1997).

[78] Glassey, R. T., and Schaeffer, J., "The relativistic Vlasov-Maxwell system in two space dimensions. Part 1.", *Arch. Rat. Mech. Anal.*, **141**, 331–354, (1998).

[79] Glassey, R. T., and Schaeffer, J., "The relativistic Vlasov-Maxwell system in two space dimensions. Part 2.", *Arch. Rat. Mech. Anal.*, **141**, 355–374, (1998).

[80] Glassey, R. T., and Strauss, W., "Asymptotic stability of the relativistic Maxwellian.", *Publ. Math. RIMS Kyoto*, **29**, 167–233, (1993).

[81] Glimm, J., "Solutions in the large for nonlinear hyperbolic systems of equations.", *Commun. Pure Appl. Math.*, **18**, 697–715, (1965).

[82] Goliath, M., Nilsson, U., and Uggla, C., "Spatially self-similar spherically symmetric perfect-fluid models.", *Class. Quantum Grav.*, **15**, 167–186, (1998).

[83] Goliath, M., Nilsson, U., and Uggla, C., "Timelike self-similar spherically symmetric perfect-fluid models.", *Class. Quantum Grav.*, **15**, 2841–2863, (1998).

[84] Grassin, M., "Global smooth solutions to Euler equations for a perfect gas.", *Indiana Univ. Math. J.*, **47**, 1397–1432, (1998).

[85] Gundlach, C., "Critical phenomena in gravitational collapse.", *Adv. Theor. Math. Phys.*, **2**, 1–49, (1998).

[86] Guo, Y., "Smooth irrotational flows in the large to the Euler-Poisson system.", *Commun. Math. Phys.*, **195**, 249–265, (1998).

[87] Guo, Y., and Rein, G., "Stable steady states in stellar dynamics.", to appear in Arch. Rat. Mech. Anal.

[88] Guo, Y., and Tahvildar-Zadeh, A. S., "Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics", in Chen, G.-Q., and DiBenedetto, E., eds., *Nonlinear partial differential equations*, (AMS, Providence, 1999).

[89] Hartman, P., *Ordinary Differential Equations*, (Birkhäuser, Basel, 1982).

[90] Hauser, I., and Ernst, F. J., "Group structure of the solution manifold of the hyperbolic Ernst equation - general study of the subject and detailed elaboration of mathematical proofs", (March, 1999), [Online Los Alamos Archive Preprint]: cited on 4 December 1999, http://xxx.lanl.gov/abs/gr-qc/9903104.

[91] Heilig, U., "On the existence of rotating stars in general relativity.", *Commun. Math. Phys.*, **166**, 457–493, (1995).

[92] Heusler, M., "Stationary black holes: uniqueness and beyond", (1998).

[93] Heusler, M., *Black hole uniqueness theorems.*, (Cambridge University Press, Cambridge, 1996).

[94] Hubbard, J. H., and West, B. H., *Differential equations: a dynamical systems approach.*, (Springer, Berlin, 1991).

[95] Isenberg, J., "Constant mean curvature solutions of the Einstein constraint equations on closed manifolds.", *Class. Quantum Grav.*, **12**, 2249–2274, (1995).

[96] Isenberg, J., and Kichenassamy, S., "Asymptotic behaviour in polarized T^2-symmetric vacuum spacetimes", *J. Math. Phys.*, **40**, 340–352, (1999).

[97] Isenberg, J., and Moncrief, V., "Asymptotic behaviour of the gravitational field and the nature of singularities in Gowdy spacetimes.", *Ann. Phys.(NY)*, **199**, 84–122, (1990).

[98] Isenberg, J., and Moncrief, V., "A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds.", *Class. Quantum Grav.*, **13**, 1819–1847, (1996).

[99] Isenberg, J., and Rendall, A. D., "Cosmological spacetimes not covered by a constant mean curvature slicing.", *Class. Quantum Grav.*, **15**, 3679–3688, (1998).

[100] John, F., *Partial differential equations*, (Springer, Berlin, 1982).

[101] John, F., *Nonlinear wave equations, formation of singularities*, (American Mathematical Society, Providence, 1990).

[102] Kichenassamy, S., "The blow-up problem for exponential nonlinearities", *Commun. P. D. E.*, **21**, 125–162, (1996).

[103] Kichenassamy, S., "Fuchsian equations in Sobolev spaces and blow-up", *J. Diff. Eq.*, **125**, 299–327, (1996).

[104] Kichenassamy, S., *Nonlinear wave equations*, (Marcel Dekker, New York, 1996).

[105] Kichenassamy, S., and Littman, W., "Blow-up surfaces for nonlinear wave equations, I", *Commun. P. D. E.*, **18**, 431–452, (1993).

[106] Kichenassamy, S., and Littman, W., "Blow-up surfaces for nonlinear wave equations, II", *Commun. P. D. E.*, **18**, 1869–1899, (1993).

[107] Kichenassamy, S., and Rendall, A. D., "Analytic description of singularities in Gowdy spacetimes", *Class. Quantum Grav.*, **15**, 1339–1355, (1998).

[108] Kind, S., and Ehlers, J., "Initial boundary value problem for the spherically symmetric Einstein equations for a perfect fluid", *Class. Quantum Grav.*, **18**, 2123–2136, (1993).

[109] Klainerman, S., and Machedon, M., "Finite energy solutions of the Yang-Mills equations in $\mathbf{R}^{3+1}$.", *Ann. Math.*, **142**, 39–119, (1995).

[110] Klainerman, S., and Nicoló, F., "On local and global aspects of the Cauchy problem in general relativity.", *Class. Quantum Grav.*, **16**, R73–R157, (1999).

[111] Kleihaus, B., and Kunz, J., "Static axially symmetric Einstein-Yang-Mills-dilaton solutions: I. Regular solutions.", *Phys. Rev. D*, **57**, 834–856, (1998).

[112] Lifshitz, E. M., and Khalatnikov, I. M., "Investigations in relativistic cosmology.", *Adv. Phys.*, **12**, 185–249, (1963).

[113] Lin, S. S., "Stability of gaseous stars in spherically symmetric motions.", *SIAM J. Math. Anal.*, **28**, 539–569, (1997).

[114] Lindblom, L., and Masood-ul Alam, A. K. M., "On the spherical symmetry of static stellar models.", *Commun. Math. Phys.*, **162**, 123–145, (1994).

[115] Lions, P.-L., "Compactness in Boltzmann's equation via Fourier integral operators and applications.", *J. Math. Kyoto Univ.*, **34**, 391–427, (1994).

[116] Lions, P.-L., and Perthame, B., "Propagation of moments and regularity for the three-dimensional Vlasov-Poisson system.", *Invent. Math.*, **105**, 415–430, (1991).

[117] Majda, A., *Compressible fluid flow and systems of conservation laws in several space dimensions.*, (Springer, Berlin, 1984).

[118] Makino, T., "On spherically symmetric stellar models in general relativity.", *J. Math. Kyoto Univ.*, **38**, 55–69, (1998).

[119] Moncrief, V., "Global properties of Gowdy spacetimes with $T^3 \times R$ topology.", *Ann. Phys.(NY)*, **132**, 87–107, (1981).

[120] Moncrief, V., "Neighbourhoods of Cauchy horizons in cosmological spacetimes with one Killing field", *Ann. Phys.(NY)*, **141**, 83–103, (1982).

[121] Moncrief, V., and Eardley, D., "The global existence problem and cosmic censorship in general relativity.", *Gen. Rel. Grav.*, **13**, 887–892, (1981).

[122] Mucha, P. B., "Global existence for the Einstein-Boltzmann equation in flat Robertson-Walker spacetime", *Commun. Math. Phys.*, **203**, 107–118, (1999).

[123] Newman, R. P. A. C., "On the structure of conformal singularities in classical general relativity.", *Proc. R. Soc. London*, **443**, A473–492; A493–515, (1993).

[124] Park, J., "Static solutions of the Einstein equations for spherically symmetric elastic bodies", (October, 1998), [Online Los Alamos Archive Preprint]: cited on 17 August 1999, http://xxx.lanl.gov/abs/gr-qc/9810010.

[125] Pfaffelmoser, K., "Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data.", *J. Diff. Eq.*, **95**, 281–303, (1992).

[126] Racke, R., *Lectures on nonlinear evolution problems.*, (Vieweg, Wiesbaden, 1992).

[127] Rein, G., "Stationary and static stellar dynamical models with axial symmetry.", to appear in Journal of Nonlinear Analysis.

[128] Rein, G., "Generic global solutions of the relativistic Vlasov-Maxwell system of plasma physics.", *Commun. Math. Phys.*, **135**, 41–78, (1990).

[129] Rein, G., "Static solutions of the spherically symmetric Vlasov-Einstein system.", *Math. Proc. Camb. Phil. Soc.*, **115**, 559–570, (1994).

[130] Rein, G., "Cosmological solutions of the Vlasov-Einstein system with spherical, plane and hyperbolic symmetry.", *Math. Proc. Camb. Phil. Soc.*, **119**, 739–762, (1996).

[131] Rein, G., "Nonlinear stability of homogeneous models in Newtonian cosmology.", *Arch. Rat. Mech. Anal.*, **140**, 335–351, (1997).

[132] Rein, G., and Rendall, A. D., "Compact support of spherically symmetric equilibria in relativistic and non-relativistic galactic dynamics", (December, 1998), [Online Los Alamos Archive Preprint]: cited on 17 August 1999, http://xxx.lanl.gov/abs/gr-qc/9812061.

[133] Rein, G., and Rendall, A. D., “Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data”, *Commun. Math. Phys.*, **150**, 561–583, (1992).

[134] Rein, G., and Rendall, A. D., “Smooth static solutions of the spherically symmetric Vlasov-Einstein system.”, *Ann. Inst. H. Poincaré (Physique Théorique)*, **59**, 383–397, (1993).

[135] Rein, G., and Rendall, A. D., “Global existence of classical solutions to the Vlasov-Poisson system in a three dimensional, cosmological setting.”, *Arch. Rat. Mech. Anal.*, **126**, 183–201, (1994).

[136] Rein, G., Rendall, A. D., and Schaeffer, J., “A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system.”, *Commun. Math. Phys.*, **168**, 467–478, (1995).

[137] Rein, G., Rendall, A. D., and Schaeffer, J., “Critical collapse of collisionless matter - a numerical investigation.”, *Phys. Rev. D*, **D58**, 044007, (1998).

[138] Rendall, A. D., “Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations.”, *Proc. R. Soc. Lond.*, **A427**, 221–239, (1990).

[139] Rendall, A. D., “The initial value problem for a class of general relativistic fluid bodies.”, *J. Math. Phys.*, **33**, 1047–1053, (1992).

[140] Rendall, A. D., “On the definition of post-Newtonian approximations.”, *Proc. R. Soc. Lond.*, **A438**, 341–360, (1992).

[141] Rendall, A. D., “The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system.”, *Commun. Math. Phys.*, **163**, 89–112, (1994).

[142] Rendall, A. D., “Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry.”, *Class. Quantum Grav.*, **12**, 1517–1533, (1995).

[143] Rendall, A. D., “Constant mean curvature foliations in cosmological spacetimes.”, *Helv. Phys. Acta*, **69**, 490–500, (1996).

[144] Rendall, A. D., “Existence and non-existence results for global constant mean curvature foliations.”, *Nonlinear Analysis, Theory, Methods and Applications*, **30**, 3589–3598, (1997).

[145] Rendall, A. D., “Existence of constant mean curvature hypersurfaces in spacetimes with two-dimensional local symmetry.”, *Commun. Math. Phys.*, **189**, 145–164, (1997).

[146] Rendall, A. D., "Global dynamics of the mixmaster model.", *Class. Quantum Grav.*, **14**, 2341–2356, (1997).

[147] Rendall, A. D., "An introduction to the Einstein-Vlasov system", *Banach Centre Publications*, **41**, 35–68, (1997).

[148] Rendall, A. D., "Solutions of the Einstein equations with matter.", in Francaviglia, M., Longhi, G., Lusanna, L., and Sorace., E., eds., *Proceedings of the 14th International Conference on General Relativity and Gravitation.*, 313–335, (World Scientific, Singapore, 1997).

[149] Rendall, A. D., and Schmidt, B. G., "Existence and properties of spherically symmetric static fluid bodies with given equation of state.", *Class. Quantum Grav.*, **8**, 985–1000, (1991).

[150] Rendall, A. D., and Tod, K. P., "Dynamics of spatially homogeneous solutions of the Einstein-Vlasov equations which are locally rotationally symmetric", *Class. Quantum Grav.*, **16**, 1705–1726, (1999).

[151] Ringström, H., "Curvature blow up in Bianchi VIII and IX vacuum spacetimes", (November, 1999), [Online Los Alamos Archive Preprint]: cited on 4 December 1999, http://xxx.lanl.gov/abs/gr-qc/9911115.

[152] Schaeffer, J., "A class of counterexamples to Jeans' theorem for the Vlasov-Einstein system.", *Commun. Math. Phys.*, **204**, 313–327, (1999).

[153] Secchi, P., "On the equations of viscous gaseous stars.", *Ann. Scuola Norm. Pisa*, **18**, 295–318, (1991).

[154] Shapiro, S. L., and Teukolsky, S. A., "Relativistic stellar dynamics on the computer. II Physical applications.", *Astrophys. J.*, **298**, 58–79, (1985).

[155] Sideris, T., "Formation of singularities in three-dimensional compressible fluids.", *Commun. Math. Phys.*, **101**, 475–485, (1979).

[156] Smoller, J., and Temple, B., "Global solutions of the relativistic Euler equations.", *Commun. Math. Phys.*, **156**, 65–100, (1993).

[157] Smoller, J. A., Wasserman, A. G., Yau, S. T., and Mcleod, J. B., "Smooth static solutions of the Einstein-Yang-Mills equations.", *Commun. Math. Phys.*, **143**, 115–147, (1991).

[158] Strauss, W., *Nonlinear wave equations*, (American Mathematical Society, Providence, 1989).

[159] Taylor, M. E., *Pseudodifferential operators and nonlinear PDE.*, (Birkhäuser, Boston, 1991).

[160] Taylor, M. E., *Partial differential equations.*, (Springer, Berlin, 1996).

[161] Tod, K. P., "Isotropic singularities.", *Rend. Sem. Mat. Univers. Politecn. Torino*, **50**, 69–93, (1992).

[162] Wainwright, J., and Ellis, G. F. R., *Dynamical Systems in Cosmology.*, (Cambridge University Press, Cambridge, 1997).

[163] Wainwright, J. A., Hancock, M. J., and Uggla, C., "Asymptotic self-similarity breaking at late times in cosmology.", *Class. Quantum Grav.*, **16**, 2577–2598, (1999).

[164] Weaver, M., "Dynamics of magnetic Bianchi VI_0 cosmologies.", (September, 1999), [Online Los Alamos Archive Preprint]: cited on 21 September 1999, http://xxx.lanl.gov/abs/gr-qc/9909043.

[165] Witt, D., "Vacuum spacetimes that admit no maximal slice.", *Phys. Rev. Lett.*, **57**, 1386–1389, (1986).

[166] Woodhouse, N. M. J., "Integrability and Einstein's equations.", *Banach Center Publications*, **41, 1**, 221–232, (1997).

[167] Wu, S., "Well-posedness in Sobolev spaces of the full water wave problem in 3-D", *J. Amer. Math. Soc.*, **12**, 445–495, (1999).

[168] York, J. W., "Conformal 'thin-sandwich' data for the initial value problem of general relativity.", *Phys. Rev. Lett.*, **82**, 1350–1353, (1999).